Special metrics in Complex Geometry

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Abstract.

In the first part of my talk, we consider special metrics on holomorphic bundles. We will recall the classical Hitchin-Kobayashi correspondence (Donaldson-Uhlenbeck-Yau theory) of stability and Hermitian-Einstein metrics on holomorphic vector bundles; and some generalizations of the classical Hitchin-Kobayashi correspondence, specially, we will focus on non-compact case; furthermore, We'll discuss the Dirichlet boundary problem of Hermitian-Einstein equations (or quiver Vortex equations) and some related heat flow in gauge theory.

In the second part of my talk, also I'll introduce a uniqueness result about constant σ_k curvature Kähler metrics.

1 Hitchin-Kobayashi type correspondences (compact case)

1.1 The classical Hitchin-Kobayashi correspondence

Let (M, η) be a compact Kähler manifold, η is the Kähler form, and let E be a holomorphic vector bundle over M.

The stability of holomorphic vector bundles was a well established concept in algebraic geometry. A holomorphic vector bundle E is called **stable** , if for every coherent sub-sheaf $E' \hookrightarrow E$ of lower rank (or, weakly holomorphic sub-bundle) it holds $\mu(E') < \mu(E)$. Where the η-slope of E' is the quotient

$$\mu(E') = \frac{deg_\eta(E')}{rank E'}, \tag{1.1}$$

and the η-degree of E' is defined as follow

$$deg_\eta(E') = \int_M C_1(E') \wedge \eta^{n-1}, \tag{1.2}$$

$C_1(E')$ is the first chern class of E'

Hermitian-Einstein metric this notion was introduced by Kobayashi in 1980 in any arbitrary holomorphic vector bundle over a complex manifold, which can be seen as a generalization of a Kähler-Einstein metric in the tangent bundle.

A Hermitian metric H in E is called a Hermitian-Einstein metric, if the curvature F_H of the chern connection A_H in (E, H) (i.e. the unique H-unitary integrable connection A_H in E inducing the holomorphic structure $\bar{\partial}_E$) satisfies the Einstein condition:

$$\sqrt{-1}\Lambda_\eta F_H = \gamma Id_E, \tag{1.3}$$

here Λ_η denotes the contraction of differential forms by Kähler form η, and the real constant γ is given by $\gamma = \frac{2\pi}{(n-1)! Vol(M)}\mu(E)$.

A H-unitary connection A is called Hermitian-Einstein if it's curvature 2-form F_A is of type (1,1), and satisfies the above Einstein condition

The classical Hitchin-Kobayashi correspondence states that a holomorphic structure is stable if and only if it is simple (i.e. it admits no non-trivial trace free infinitesimal automorphisms) and admits a Hermitian-Einstein metric. General solutions of the Hermitian-Einstein equation correspond to polystable holomorphic structure, i.e. to bundles which are direct sum of stable bundle of the same slope.

A short history of the proof of the Hitchin-Kobayashi correspondence.

Narasimhan and Seshadri ([NS], 1965) , Donaldson ([Do1], 1983,), for Riemann surface.

Donaldson ([Do2],1985) for algebraic surfaces.

Uhlenbeck and Yau ([UY], 1986) for Kähler manifolds.

Li and Yau ([LY], 1987) for Hermitian manifolds with Gauduchon metric (i.e. $\partial\bar{\partial}(\eta^{n-1}) = 0$).

1.2 Some generalizations of the classical Hitchin-Kobayashi correspondence

The classical Hitchin-Kobayashi correspondence has several interesting and important generalization and extensions. Below we sketch three problems of this type, and in every case we briefly explain what the corresponding Hitchin-Kobayashi correspondence asserts.

1, Higgs bundle

Higgs bundles were first studied by Hitchin [Hi] when M is a compact Riemann surface and Simpson [Si] when M is higher dimensional, who introduce a natural gauge equation for them and proved a Hitchin-Kobayashi correspondence.

A Higgs bundle on M is a pair (E, θ) consisting of a holomorphic vector bundle E, and an $End(E)$-valued holomorphic form $\theta \in H^0(M, \Omega^1 \otimes End(E))$ satisfying the identity $\theta \wedge \theta = 0$.

A Higgs bundle is called **Stable** if the usual stability condition $(\mu(E') < \mu(E))$ hold for all proper θ-invariant sub-sheaves; and is called polystable if it is a direct sum of stable Higgs sub-bundle of the same slope.

A Hermitian metric H in Higgs bundle (E, θ) is called **Hermitian-Einstein** (or Hermitian Yang-Mills) if the curvature F of the (in general non-integrable and non-unitary) connection $A = \bar{\partial}_E + \theta + \partial_H + \bar{\theta}$ satisfies the Einstein condition $\sqrt{-1}\Lambda F = cId_E$.

The Hitchin-Kobayashi correspondence for Higgs bundles asserts that: a Higgs bundle admits a Hermitian-Einstein metric if and only if it is polystable.

2, Vortices (holomorphic pair).

Different to the Higgs bundle, Bradlow [Br1], [Br2] consider holomorphic vector bundles on which additional data in the form of a prescribed holomorphic global section $\phi \in \Gamma(E)$ is given, i.e. holomorphic pair (E, ϕ). Bradlow investigate the follow **vortex equation**

$$2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^* - \tau Id_E = 0, \tag{1.4}$$

where ϕ^* is the adjoint of ϕ with respect to metric H, and τ is a real number. This equation generalizes the Hermitian-Einstein equation and is the analog of the classical vortex equation over R^2. Equivalently, on Hermitian vector bundle (E, H) we can consider vortices, i.e. pairs (A, ϕ) consisting of an integrable unitary connection A and a holomorphic section ϕ, satisfying the vortex equation

$$2\sqrt{-1}\Lambda F_A + \phi \otimes \phi^* - \tau Id_E = 0.$$

The holomorphic pair (E, ϕ) is called τ**-stable** if it satisfies the following two conditions:

(1), It holds $\frac{2\pi}{(n-1)!Vol(M)}\mu(E) < \tau$, and $\frac{2\pi}{(n-1)!Vol(M)}\mu(E') < \tau$ for all reflexive sub-sheaves $E' \hookrightarrow E$ with $0 < rank(E') < rankE$.

(2), $\frac{r\mu(E)-r'\mu(E')}{r-r'} > \frac{\tau Vol(M)}{4\pi}$, for every reflexive sub-sheaves E' with $0 < rankE' = r' < r$ such that $\phi \in E'$ almost everywhere.

The Hitchin-Kobayashi correspondence asserts that there exist a Hermitian metric satisfying τ-vortex equation if and only if the holomorphic pair (E, ϕ) is τ-poly-stable.

(3), Holomorphic triple

Holomorphic triple (E_1, E_2, ϕ)consisting of two holomorphic vector bundles E_1, E_2 over complex manifold M and a holomorphic morphism $\phi : E_2 \to E_1$ (i.e. $\bar{\partial}_{E_1 \otimes E_2^*}\phi = 0$). The coupled vortex equation, as a generalization of the vortex equations, were introduced by Garcia-Prada [GP1]([BG]). The equations we shall consider are

$$\{ \begin{array}{c} \sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^* - \tau_1 Id_{E_1} = 0, \\ \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^* \circ \phi - \tau_2 Id_{E_2} = 0. \end{array} \tag{1.5}$$

Where H_1 and H_2 are Hermitian metrics on bundles E_1, E_2 respectively.

Definition 1.1 *A triple $T' = (E_1', E_2', \phi')$ is a sub-triple of (E_1, E_2, ϕ) if*

(1), E_i' is a coherent sub-sheaf of E_i, for $i = 1, 2$.
(2), $i_1 \circ \phi' = \phi \circ i_2$, where i_1 and i_2 are inclusion maps.

Definition 1.2 *Let σ (in fact, $\sigma = \frac{2\pi}{Vol(M)}\tau_1$) be a real number, define the σ-degree and σ-slope of a subtriple $T' = (E_1', E_2', \phi')$ by*

$$deg_\sigma(T') = deg(E_1' \oplus E_2') + r_2'\sigma, \tag{1.6}$$

$$\mu_\sigma(T') = \frac{deg_\sigma(T')}{r_1' + r_2'}.$$

Definition 1.3 *The triple $T = (E_1, E_2, \phi)$ is called σ-**stable** if for all non-trivial sub-triples T' we have $\mu_\sigma(T') < \mu_\sigma(T)$.*

Bradlow and Garcia-Prada [BG] proved the following Hitchin-Kobayashi type correspondence.

The holomorphic triple exists a solution of the coupled vortex equations if and only if it is σ-stable, in fact, $\sigma = \frac{2\pi}{Vol(M)}\tau_1$.

Holomorphic pair can be seen as a special case of holomorphic triple, i.e. let E_2 be a line bundle L on M, $(E, L, \phi) = (E, \phi)$. In [GP1], Garcia-prada show that the above coupled vortex equation can also be obtained via dimensional reduction of classical Hermitian-Einstein equations under an $SU(2)$ action on certain associated bundles on the manifold $M \times CP^1$ $(\eta \otimes \sigma\omega_{CP^1})$. Recently, Garcia-prada's idea has been used by Tian and Yang to discuss the compactification of the moduli space of vortices and coupled vortices.

Recently, the above results of holomorphic triple had been extended to **holomorphic chain** case by Alvare-consal and Garcia-prada [AG1]. Furthermore, their result has been extended by Mundet i Riera [MR] to more general Kähler fibration, and by Alvare-Consul and Garcia-Prada [AG2],[AG3] to twisted quiver bundles. In [Zh1], we obtained a Hitchin-Kobayashi type for twisted quiver bundles over hermitian manifolds with Gauduchon metrics.

(4), Twisted quiver bundle

Twisted quiver bundles over Kähler manifolds were studied by Alvare-Consul and Garcia-Prada [AG2],[AG3].

A **quiver** Q consists of a set Q_0 of vertices v, v',$\cdots$, and a set Q_1 of arrows $a : v \to v'$ connecting the vertices. Given a quiver Q and a compact Kähler manifold M, a **quiver bundle** is defined by assigning a holomorphic vector bundle E_v to a finite number of vertices and a homomorphism $\phi_a : E_v \to E_{v'}$ to a finite number of arrows. A *quiver sheaf* is defined by replacing the term "holomorphic vector bundle " by "coherent sheaf" in the above definition. If we fix a collection of holomorphic vector bundles $\tilde{E}_a$ parametrized by the set of arrows, and the morphisms are $\phi_a : E_v \otimes \tilde{E}_a \to E_{v'}$, twisted by the corresponding bundles, we have a **twisted quiver bundle** or a *twisted quiver sheaf.*

In [AG2] Alvare-Consul and Garcia-Prada defined natural gauge-theoretic equations, **quiver vortex equations**, for a collection of Hermitian metrics on the bundles associated to the vertices of a twisted quiver bundle. To solve these equations, they introduced a stability criterion for twisted quiver sheaves, and proved a **Hitchin-Kobayashi correspondence**, relating the existence of Hermitian metrics satisfying the quiver vortex equations to the stability bundle.

The above result generalized many known results for bundles with extra structure. For examples: *Higgs bundles*, *holomorphic pair* , *holomorphic triple, holomorphic chain* . It should be pointed out Alvare-Consul and Garcia-Prada's results ([AG2],[AG3]) can not be derived from the general Hitchin-Kobayashi correspondence scheme developed by Banfield [5] and further generalized by Mundet i Riera [MR]. This is due not oly to the presence of twisting vector bundles, but also to the deformation of the Hermitian-Einstein terms in the equations.

2 Hitchin-Kobayashi correspondence over non-compact Kähler manifolds

In this section, we will give a Hitchin-Kobayashi type correspondence for holomorphic triples over some non-compact Kähler manifolds. Our method can also be suited in other more general cases (for example twisted quiver bundle), but for simplicity, we only discuss the coupled vortex equations

$$\left\{ \begin{array}{l} \sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^* - \tau_1 Id_{E_1} = 0, \\ \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^* \circ \phi - \tau_2 Id_{E_2} = 0. \end{array} \right. \tag{2.1}$$

Holomorphic triple consisting of two holomorphic vector bundles E_1, E_2 over complex manifold M and a holomorphic morphism $\phi : E_2 \to E_1$ (i.e. $\bar{\partial}_{E_1 \otimes E_2^*}\phi = 0$), where H_1 and H_2 are Hermitian metrics on bundles E_1, E_2 respectively.

It should be point out that Simpson [Si] had discussed Higgs bundle over non-compact cases under some conditions, Li,jianyu and Wang,youde [LW] generalized Simpson's result under more weaker assumption.

Let M be an non-compact Riemannian manifold satifying the following assumptions:

Assumption 1. *M has finite volume.*

Assumption 2. *There exists an exhaustion non-negative function φ on M with $\triangle\varphi$ bounded.*

Assumption 3. *There is an increasing function $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$ and $\alpha(x) = x$ for $x > 1$,such that if f is a bounded positive function on M with $\triangle f \geq -B$ then*

$$\sup_M |f| \leq C(B)\alpha(\int_M |f|).$$

Simpson [Si, Proposition 2.1 and Proposition 2.2] proved that if (M,ω) is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ (i.e. $\overline{M}-M$ is a smooth divisor)and the metric ω is the restriction of a smooth Kähler metric on $\overline{M}$, then the above assumption hold for (M,ω).

We will use the heat flow method to solve coupled vortex (2.1) which is similar with that used by Simpson in Higgs bundles case, but the main different point in our proof is to find good initial metrics on holomorphic triples by using **conformal transformation**, this is not trivial when the Kähler manifold is non-compact. In fact, we can find Hermitian metrics K_1 on bundle E_1 and K_2 on bundle E_2 by conformal transformation such that

$$Tr(\sqrt{-1}\Lambda F_{K_1}) + Tr(\sqrt{-1}\Lambda F_{K_2}) = Constant. \tag{2.2}$$

If $H = e^f K$, we have $2Tr(\sqrt{-1}\Lambda F_H) = 2Tr(\sqrt{-1}\Lambda F_K) - n\triangle f$. In order to obtain the above good initial Hermitian metrics, we shall first solve the following Kazdan-Warner equations over non-compact Riemannian manifolds.

2.1 The Kazdan-Warner type equations on non-compact Riemannian manifolds

Let M be a Riemannian manifold, and let $h, g \in C^\infty(M)$, we consider the following equation on M,

$$\triangle f + he^f - g = 0. \tag{2.3}$$

Equation (2.3) is almost in the form of a non-linear PDE analysed by Kazdan and Warner [KW1]. Where they want to find Riemannian metrics on a compact manifold such that its Gaussian curvature is a given function. When M be a **compact** Riemannian manifold (without boundary), we can solve the following Poisson equation,

$$\triangle v = g - c, \tag{2.4}$$

where $c = \frac{\int_M g}{Vol(M)}$. Define

$$w = f - v.$$

Then f is a solution to (2.3) iff w is a solution to

$$\triangle w + (he^v)e^w - c = 0. \tag{2.5}$$

Theorem 2.1 (Kazdan-Warner) *Let M be a compact Riemannian manifold. Consider the equation*

$$\triangle u + he^u - c = 0 \tag{2.6}$$

where $h \in C^\infty(M)$ is not identically zero and c is a real constant. Then

(1), if $c = 0$ a necessary condition for existence of $u \in C^\infty(M)$ satisfying (2.6) is that h change sign on M.

(2), if $c > 0$ a necessary condition for existence of $u \in C^\infty(M)$ satisfying (2.6) is that h is strictly positive somewhere on M.

(3), if $c < 0$ and $h \leq 0$ then there is a unique $u \in C^\infty(M)$ which satisfies (2.6).

Kazdan and Warner [KW2]also discussed the above equation on all non-compact 2-dimensional manifolds realizable as open sub-manifolds as compact 2-manifolds. Now, we want to consider the equation (2.3) on some non-compact higher dimensional Riemannian manifolds.

Theorem 2.2 [Zh2] *Let M be a non-compact Riemannian manifold satisfying the above assumption 1,2,3. We consider the equation*

$$\triangle f + he^f - g = 0. \tag{2.3}$$

and suppose that $h, g \in C^\infty(M)$ and $\sup_M(|g| + |h|) \leq \infty$. Then

(1), if $h \equiv 0$ and $\int_M g = 0$, i.e. the equation (2.3) is just the Poisson equation, there exists $f \in C^\infty(M)$ satisfying (2.3) and $\sup_M |f| < \infty$.

(2), if $h \leq 0$ is not identically zero and $\int_M g < 0$, then there is a unique $f \in C^\infty(M)$ which satisfies (2.3) and $\sup_M |f| < \infty$.

Remark: *We use the heat flow method to prove the above theorem. We first solve the heat equation*

$$\frac{\partial f}{\partial t} = \triangle f + he^f - g$$

satisfying the Neumann boundary condition with initial data 0 on any exhaustion subset M_a of M, then using the exhaustion method we can prove there is a longtime solution $f(\cdot, t)$ of the heat flow on M.

the main point in our proof is to obtain an uniform C^0 bound for $f(\cdot, t)$, then we can choose a subsequence $t_j \rightarrow$ such that $f(\cdot, t_j) \rightarrow f_\infty$ where f_∞ is just a solution of (2.3).

Using the above theorem we can solve the following vortex equations on holomorphic linear bundle L over some non-compact Kähler manifolds (M, η).

$$2\sqrt{-1}\Lambda F_H + \phi \otimes \phi^* - \tau Id_L = 0, \tag{2.7}$$

where ϕ is a nontrivial holomorphic section L, ϕ^* is the adjoint of ϕ with respect to metric H, and τ is a real number, here Λ_η denotes the contraction of differential forms by Kähler form η.

When L is a linear holomorphic bundle, given any initial metric K, let $H = e^f K$, then solving the Vortex equation (2.7) is equivalent to solve the following

$$\triangle f - |\phi|_K^2 e^f - (2\sqrt{-1}\Lambda F_K - \tau Id_L) = 0. \tag{2.8}$$

Setting $h = -|\phi|_K^2$ and $2\sqrt{-1}\Lambda F_K - \tau Id_L = g$, (2.8) is just equation (2.3).

Using Theorem 2.2, we have

Corollary 2.3 *Let (M, ω) satisfies the above assumption 1,2,3,(for example a Zariski open subset of a smooth compact Kähler manifold) and L be a linear holomorphic bundle with a nontrivial holomorphic section ϕ on M. Suppose that there exist a Hermitian metric* $\mathbf{K}$ *satisfying that $\sup_M |\Lambda F_K| < \infty$, $\sup_M |\phi|_K^2 \leq \infty$ and $\int_M 2\sqrt{-1}\Lambda F_K < \tau$. Then there is a Hermitian metric H satisfies the vortex equations (2.7).*

2.2 Coupled Vortex equations over non-compact Kähler manifolds

Then, we consider the following heat equation

$$\begin{cases} H_1^{-1}\frac{\partial H_1}{\partial t} & = -2(\sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^* - \tau_1 Id_{E_1}), \\ H_2^{-1}\frac{\partial H_2}{\partial t} & = -2(\sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^* \circ \phi - \tau_2 Id_{E_2}). \end{cases} \tag{2.9}$$

And let the good metrics (K_1, K_2) be the initial datas. Then we can obtain a longtime solution (H_1, H_2) of the above heat flow, under the analytic stable condition, we can prove the longtime solution must convergence to a solution of coupled vortex equation (2.1) by choosing a subsequence. So we obtain

Theorem 2.4 *Let (M, ω) satisfies the above assumption 1,2,3, and suppose (E_1, E_2, ϕ) is a holomorphic triple with a 2-tuple of Hermitian metrics (K_1, K_2) satisfying the assumption that $\sup_M(\sum_{i=1}^2 |\Lambda F_{K_i}| + |\phi|_K) < \infty$. Suppose (E_1, E_2, ϕ) is analytic stable with respect to metrics K_1, K_2. Then there is a 2-tuple of Hermitian metrics (H_1, H_2) satisfy the coupled vortex equations*

$$\begin{cases} \sqrt{-1}\Lambda F_{H_1} + \frac{1}{2}\phi \circ \phi^* - \tau_1 Id_{E_1} = 0, \\ \sqrt{-1}\Lambda F_{H_2} - \frac{1}{2}\phi^* \circ \phi - \tau_2 Id_{E_2} = 0. \end{cases}$$

where real numbers τ_1 and τ_2 satisfy

$$\int_M Tr(\sqrt{-1}\Lambda F_{K_1} + \sqrt{-1}\Lambda F_{K_2}) = rank(E_1)\tau_1 + rank(E_2)\tau_2$$

.

Analytic stable

A triple $T' = (E'_1, E'_2, \phi')$ is a sub-triple of (E_1, E_2, ϕ) if
(1), E'_i is a coherent sub-sheaf of E_i, for $i = 1, 2$.
(2), $i_1 \circ \phi' = \phi \circ i_2$, where i_1 and i_2 are inclusion maps.

Since E'_i is a coherent sub-sheaf of E_i then outside of complex co-dimension 2 it is a sub-bundle of E_i. The metric K_i restricts to a metric on E'_i outside complex co-dimension two. Let $\pi_i : E_i \to E'_i$ denote the projection onto E'_i using the metric K_i,it is also defined outside complex codimension two. So we can define the degree by integrating outside complex co-dimension two.

The τ-degree and τ-slope of a sub-chain $\mathbf{T}'$ with respect to metric $\mathbf{K}$, are defined by

$$\begin{array}{ll} deg_\tau(\mathbf{T}', \mathbf{K}) & = \int_M [\sum_{i=1}^2 (Tr\pi_i \circ \theta_i(K;\tau) - |\overline{\partial}_{E_i \otimes E_i^*}\pi_i|_K^2) \\ & -|\phi^\perp|_K^2]\omega^{[m]}, \\ \mu_\tau(\mathbf{T}', K) & = \frac{deg_\alpha(\mathbf{T}', K)}{\sum_{i=1}^2 rankE'_i}, \end{array}$$

respectively. Where

$$\phi^\perp = \pi_1 \circ \phi \circ (Id_{E_2} - \pi_2),$$

$$\theta_1 = \sqrt{-1}\Lambda F_{K_1} + \frac{1}{2}\phi \circ \phi^* - \tau_1 Id_{E_1},$$

$$\theta_2 = \sqrt{-1}\Lambda F_{K_2} - \frac{1}{2}\phi^* \circ \phi - \tau_2 Id_{E_2}.$$

We say that the holomorphic triple $\mathbf{T} = (E_1, E_2, \phi)$ is analytic τ-(semi) stable with respect to metric K if for all proper sub-triple $\mathbf{T}' \hookrightarrow \mathbf{T}$,

$$\mu_\tau(\mathbf{T}', \mathbf{K}) < (\leq)\mu_\tau(\mathbf{T}, \mathbf{K}).$$

3 Dirichlet problem for Hermitian-Einstein equations

Let M be the interior of compact Hermitian manifold $\overline{M}$ with smooth non-empty boundary ∂M, and the Hermitian metric is smooth and non-degenerate on the boundary. holomorphic bundle E is defined over $\overline{M}$. We consider the following Dirichlet problem for any given data φ on ∂M.

$$\begin{cases} \sqrt{-1}\Lambda F_H = \lambda Id, \\ H|_{\partial M} = \varphi. \end{cases} \tag{3.1}$$

we obtain the unique solubility of the above Dirichlet problem for Hermitian-Einstein equations.

Theorem 3.1[Zh3] *Assume that E is a holomorphic bundle over the compact Hermitian manifold $\bar{M}$ with non-empty boundary ∂M For any Hermitian metric φ on the restriction of E to ∂M there is a unique Hermitian-Einstein metric H on E such that $H = \varphi$ over ∂M.*

Donaldson(1992) had solved the above Dirichlet problem over Kähler manifolds. So, our theorem can be seen as a generalization of Donaldson's result.

Recently, We [Zh4] solved the Dirichlet problem for Vortex equation, and Wang [Wang1] for coupled Vortex equation. In fact, by similar discussion, we can also solve the Dirichlet problem for more general cases (for example the related gauge equations for quiver bundles over Hermitian manifolds.)

4 some related heat flows in gauge theory

Given a vector bundle E over a closed (i.e. compact and without boundary) Riemannian manifold (M, g), suppose that the bundle E has a Riemannian structure. The Yang-Mills functional is defined on the space of connections of E, here all connections are required to be compatible with the Riemannian structure of E, as follows

$$YM(A) = \int_M |F_A|^2 \, dV_g, \tag{1}$$

where A is a connection and F_A denotes its curvature and dV_g is the volume form of g. We call A a Yang-Mills connection of E if A is a critical point of the Yang-Mills functional i.e. satisfies the Yang-Mills equation

$$D_A^* F_A = 0, \tag{2}$$

where D_A^* is the adjoint operator of the covariant differentiation associated with the connection A.

The Yang-Mills-Higgs functional is defined through a connection A and a section u of the bundle E

$$YMH(A, u) = \int_M [|F_A|^2 + |D_A u|^2 + \frac{1}{4}(1 - |u|^2)^2] \, dV_g. \tag{3}$$

The Yang-Mills-Higgs fields (A, u) are the critical points for the above Yang-Mills-Higgs functional. Equivalently, the pair (A, u) satisfy the following Yang-Mills-Higgs equations:

$$\begin{cases} D_A^* F_A & = -\frac{1}{2}(D_A u \otimes u^* - u \otimes (D_A u)^*), \\ D_A^* D_A u & = \frac{1}{2} u (1 - |u|^2), \end{cases} \tag{4}$$

where u^* denote the dual of u respect to the given metric. The Yang-Mills-Higgs theory can be seen as a generalization of the Yang-Mills theory. For further discussions on its physical significance, we refer the readers to [JT].

In this section, we are interested in a more general case. Let (E_1, H_1) and (E_2, H_2) be two Riemannian vector bundles on the manifold (M, g), and let $\mathcal{A}_i$ denote the set of all connections on (E_i, H_i). consider the following Yang-Mills-Higgs type functional, which will be called Coupled Yang-Mills-Higgs functional (CYMH), on $\mathcal{A}_1 \times \mathcal{A}_2 \times \Omega^0(E_1 \otimes E_2^*)$,

$$\begin{aligned} YMH(A_1, A_2, \phi) = & \int_M |F_{A_1}|^2 + |F_{A_2}|^2 + |D_{A_1 \otimes A_2^*}\phi|^2 \\ & + \frac{1}{4}|\phi \circ \phi^* - \tau_1 Id_{E_1}|^2 + \frac{1}{4}|\phi^* \circ \phi - \tau_2 Id_{E_2}|^2 dV_g, \end{aligned} \tag{5}$$

here τ_1 and τ_2 are real parameters. We denote the integrand above by $e(A_1, A_2, \phi)$ and call it the CYMH action density for the triple (A_1, A_2, ϕ).

We call a triple (A_1, A_2, ϕ) a Coupled Yang-Mills-Higgs field if it is a critical point of the above CYMH functional, i.e. it satisfies the following equations

$$\left\{ \begin{array}{l} D_{A_1}^* F_{A_1} + \frac{1}{2}(D_{A_1 \otimes A_2^*}\phi \circ \phi^* - \phi \circ (D_{A_1 \otimes A_2^*}\phi)^*) = 0, \\ D_{A_2}^* F_{A_2} - \frac{1}{2}(\phi^* \circ D_{A_1 \otimes A_2^*}\phi - (D_{A_1 \otimes A_2^*}\phi)^* \circ \phi) = 0, \\ D_{A_1 \otimes A_2^*}^* D_{A_1 \otimes A_2^*}\phi + \phi \circ \phi^* \circ \phi - \frac{\tau_1 + \tau_2}{2}\phi = 0. \end{array} \right. \tag{6}$$

Where ϕ^* denotes the dual of ϕ with respect to the given Riemannian structures, and A_2^* denotes the induced connection on the dual bundle E_2^*. In the case of triples (E_1, E_2, ϕ) where E_2 is a line bundle, then the above Coupled Yang-Mills-Higgs equations are just the Yang-Mills-Higgs equations (setting $\tau_1 = \tau_2 = \frac{1}{2}$). So the Coupled Yang-Mills-Higgs field generalizes the Yang-Mills-Higgs field.

When (E_1, H_1) and (E_2, H_2) be Hermitian vector bundles on a compact Kähler manifold (M, ω). Consider integrable connections A_i on (E_i, H_i) $(i = 1, 2)$ and a section ϕ of $E_1 \otimes E_2^*$. The equations we shall consider are

$$\begin{cases} \overline{\partial}_{A_1\times A_2^*}\phi = 0, \\ \Lambda F_{A_1} - \frac{\sqrt{-1}}{2}\phi\circ\phi^* + \frac{\sqrt{-1}}{2}\tau_1 Id_{E_1} = 0, \\ \Lambda F_{A_2} + \frac{\sqrt{-1}}{2}\phi^*\circ\phi + \frac{\sqrt{-1}}{2}\tau_2 Id_{E_2} = 0, \end{cases} \tag{7}$$

where the operator Λ is the contraction with ω. The above equations are called the Coupled vortex equations which were introduce by Garcia-Prada in [GP], and solutions (A_1, A_2, ϕ) of them are called Coupled vortices on (E_1, E_2). By the Chern-Weil theory, τ_1 and τ_2 must satisfy the following relation

$$\tau_1 rankE_1 + \tau_2 rankE_2 = 4\pi\frac{degE_1 + degE_2}{Vol(M)}, \tag{8}$$

so that there is only one independent parameter τ_1.

In fact, Coupled vortices are the absolute minima of the above Coupled Yang-Mills-Higgs functional, so Coupled vortices must be Coupled Yang-Mills-Higgs fields. In [GP], Garcia-Prada established the Hitchin-Kobayashi correspondence between stable triples and existence of Coupled vortices. So, the existence result of Coupled Yang-Mills-Higgs fields had been obtained on Hermitian bundles.

The Yang-Mills flow was first suggested by Atiyah-Bott in [AB]. Donaldson [Do] used this to establish a connection between Hermitian-Yang-Mills connections and holomorphic stable bundles. He did this by proving the global existence of the Yang-Mills flow in a holomorphic bundle over a Kähler manifold. Global existence and uniqueness had been established by Struwe [St] for the Yang-Mills flow in a vector bundle over a compact Riemannian Four-manifold for given initial connection with finite energy. For general vector bundles, it is still unknown whether or not the Yang-Mills heat flow develops singularities in finite time. In [CS1], Chen and Shen established the monotonicity formula and the small action regularity theorem for the Yang-Mills flows in higher dimension. In [CS2], [CSZ], [DW] and [HT], the authors analyze the asymptotic behavior of the Yang-Mills flow and the Yang-Mills-Higgs flow.

In the following, we (cooperated with Wang Yue)discuss the evolution equations of the Coupled Yang-Mills-Higgs equations (6). A regular solution of the Coupled Yang-Mills-Higgs flow is given by a family of triples $(A_1(x,t), A_2(x,t), \phi(x,t))$ such that

$$\left\{ \begin{array}{lcl} \frac{\partial A_1}{\partial t} &=& -D^*_{A_1}F_{A_1} - \frac{1}{2}(D_{A_1\otimes A_2^*}\phi \circ \phi^* - \phi \circ (D_{A_1\otimes A_2^*}\phi)^*), \\ \frac{\partial A_2}{\partial t} &=& -D^*_{A_2}F_{A_2} + \frac{1}{2}(\phi^* \circ D_{A_1\otimes A_2^*}\phi - (D_{A_1\otimes A_2^*}\phi)^* \circ \phi), \\ \frac{\partial \phi}{\partial t} &=& -D^*_{A_1\otimes A_2^*}D_{A_1\otimes A_2^*}\phi - \phi \circ \phi^* \circ \phi + \frac{\tau_1+\tau_2}{2}\phi. \end{array} \right. \tag{9}$$

We first discuss some properties of the Coupled Yang-Mills-Higgs flow, including the energy inequality, Bochner-type inequality, monotonicity of certain quantities and a small action regularity theorem. Then we discuss the asymptotic behavior of a regular Coupled Yang-Mills-Higgs flow, proving the following theorem

Theorem 4.1 [WZ] *Let (E_1, H_1) and (E_2, H_2) be two Riemannian vector bundles on a compact Riemannian manifold (M, g). Let $(A_1, A_2, \phi)(x,t)$ be a global smooth solution of the Coupled Yang-Mills-Higgs flow (9) in $M \times [0, \infty)$ with smooth initial. Then there exists a sequence $\{t_i\}$ such that, as $t_i \to \infty$, $(A_1, A_2, \phi)(x, t_i)$ converges, modulo gauge transformations, to a Coupled Yang-Mills-Higgs field $(A_1, A_2, \phi)(\cdot, \infty)$ in smooth topology outside a closed set Σ whose Hausdorff codimension is at least 4.*

5 Kähler metrics with constant σ_k curvature

One of the major problems in Kähler geometry is the study of extremal metrics, especially Kähler-Einstein metrics, constant scalar curvature Kähler metrics.

Let (M, ω) be a compact Kähler manifold of complex dimension n,and assume the first Chern class of M to be negative or zero. Then by the celebrated work of Yau [Y1] (see also Aubin [A]) we know that M admits a Kähler-Einstein metric. The remaining case $c_1(M) > 0$ is more difficult and still open. In [C] Calabi proposed that, when one exists,a constant scalar curvature Kähler (cscK) metric should provide a canonical representative for a given Kähler class. Since this suggestion. much work has focused on the topic. The general existence theory has been looked at in depth, motivated by a well-known conjecture relating the existence of a cscK metric in the first Chern class of an ample line bundle L to the K-stability of the polarisation defined by L. This was first suggested by Yau [Y2] in the Kähler-Einstein case and then by Tian [T], Donaldson [D] in the cscK case. The difficulty, from the analytic viewpoint, in determining whether or not a cscK metric exists is that the resulting PDF is fourth order and fully non-linear.

It is well known ([B]) that if $c_1(M) = 0$, or if $c_1(M)$ is positive or negative definite and $\Omega = \pm 2\pi c_1(M)$, then a Kähler metric with constant scalar curvature in Kähler class Ω has to be Kähler-Einstein. In [CT], X.X.Chen and G.Tian have introduced a family of functionals E_k $(0 \leq k \leq n)$ which generalize the Mabuchi energy $\nu_\omega = E_0$, and used E_1 in their study of the Kähler-Ricci flow. The critical point of E_k are the functions ϕ such that $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ satisfies

$$\sigma_k(\omega_\phi) - \triangle_\phi \sigma_{k-1}(\omega_\phi) = \bar{\mu}_k. \tag{1}$$

Where $\sigma_k(\omega)$ is the k-th symmetric polynomial of the Ricci tensor, and $\bar{\mu}_k$ is a constant depending only on $c_1(M)$. For $k = 0$ we see that the critical metrics are precisely cscK metrics. Under the assumptions $c_1(M) > 0$ and $[\omega] = 2\pi c_1(M)$, Chen and Tian [CT] have shown that critical metrics for E_n with positive Ricci curvature must be necessarily Kähler-Einstein. In a recent paper [To] Tosatti proved that a Kähler metric ω in the anti-canonical class which is critical for E_k and has non-negative Ricci curvature,then ω is Kähler-Einstein.

In this section, we (with Guan pengfei) consider Kähler metrics with constant σ_k curvature. Under some hypotheses on orthogonal bisectional curvature, we can show that a Kähler metric with constant scalar curvature must be Kähler-Einstein. Furthermore, we conclude a similar result about Kähler metric with constantly positive σ_k curvature. In fact we obtain the following theorem.

Definition 5.1 *We say that the orthogonal bisectional curvature is nonnegative means that* $R_{i\bar{i}j\bar{j}} \geq 0$ *for any* $i \neq j$*, choosing the normal coordinate near the considered point.*

Theorem 5.2 [GZ] Let (M, ω) be a Kähler manifold , and assume that the orthogonal bisectional curvature of ω is nonnegative and is positive at least at one point. If ω is of constantly scalar curvature, then it must be Kähler-Einstein. Furthermore, fix $1 < k \leq n$ and assume that ω has constantly positive σ_k curvature, and it's Ricci curvature is nonnegative at least at one point; then it must be Kähler-Einstein.

The main point in our proof is to use the fact that $(\sigma_k)^{\frac{1}{k}}$ is concave. We known that the positivity of holomorphic bisectional curvature can deduce the positivity of orthogonal bisectional curvature and Ricci curvature. So we have the following conclusion.

Corollary 5.3 Let (M, ω) be a Kähler manifold , and assume that the holomorphic bisectional curvature of ω is nonnegative and is positive at least at one point. Fixing $1 \leq k \leq n$, if ω has constantly σ_k curvature, then it must be Kähler-Einstein.

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